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Bisingular Integral in Arithmetic Sumspaces of Summable Functions

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Abstract: Itis obtained estimates of the Zygmund estimate type for the bisingular integral. Based on the obtained estimates, it is constructed a class of functions invariant respect to thebisingular operator.

Keywords: Bisingular integral operator, Zigmund type estimate, invariant space, summable function.

1. Introduction

The classical boundedness theorem of singular operator with the Hilbert kernel in space $L_p(p > 1)$, it was proved by N.H. Luzin in [6] and M.Riesz in [16] for the cases $p = 2$ and $p > 1$, respectively.Subsequently, this result was carried over in a number of papers for fairly wide classes of Jordan rectiable curves. A detailed prehistory of this issue is available in the work [9], also in the works of A.P. Calderon [11] ,[12], and [13].

Tostudythespecialintegral

$$\tilde{u}(x) = \int_a^b \frac{u(s)}{s - x} ds\,, \qquad x \in (a, b)$$

$(-\infty < a < b < +\infty)$ with thesummable density in the work [4], [10] for a function $u \in L_p^{loc}(a, b)$, where $L_p^{loc}(a, b)$ - is the set of functions, summable with the degree p in any compact segment of the interval(a, b). The characteristics were introduced

$$\Omega_p(u, \xi, \eta) = (\int_{a+\xi}^{b-\eta} |u(x)|^p dx)^{\frac{1}{p}} \xi, \eta > 0\,, \xi + \eta \le b - a = l,$$

$$\omega_p(u, \delta, \xi, \eta) = \sup_{0<h\le\delta} (\int_{a+\xi}^{b-\eta-h} |u(x + h) - u(x)|^p dx)^{\frac{1}{p}}\,, \xi + \eta + h \le l, \delta > 0$$

and in the case $1 < p < +\infty$ it is proved estimates, $(\Omega_p(\tilde{u}), \omega_p(\tilde{u}))$, by $(\Omega_p(u), \omega_p(u))$.

In the limiting case for $p = \infty$ and $u \in C_{[a,b]}$ these results were obtained in [3], [7], it was shown that estimates [2] in a certain sense are unimprovable. In [5] using M. Riesz's theorem about the bounded action of an operator $\tilde{u}$ in the space $L_p(a, b)$ the results are obtained in [1], [2].

One of the first papers, dedicated to the repeated special integral with the Hilbert kernel

$(Bf)(x_1, x_2) = g(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + t, y + \tau) ctg \frac{t}{2} ctg \frac{\tau}{2} dt d\tau,$

was a work of L. Cesari [14]. He proved that if $f \in H^2_{(\delta_1^\alpha, \delta_2^\alpha)}$,then

$$g \in H^2_{(\delta_1^\alpha |ln\delta_1|, \delta_2^\alpha |ln\delta_2|)}$$

Following L. Cesari, I.E. Zak [5] in his work also showed that the class of functions $H^2_{(\delta_1^\alpha, \delta_2^\alpha)}$ is not invariant with respect to the operator B. In the paper, it was proved that the classes of functions

$$H^{\alpha,\beta} = \left\{f \in C_{[-\pi,\pi]^2} : \omega_f(\delta_1, \delta_2) = O\left(\delta_1^\alpha \delta_2^\beta\right), \omega_f^1(\delta_1) = O(\delta_1^\alpha), \omega_f^2(\delta_2) = O\left(\delta_2^\beta\right), 0 < \alpha, \beta < 1\right\}.$$

are invariant with respect to the operator B.

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2. RESULTS

Let $-\infty < a_1 < a_2 < +\infty, -\infty < b_1 < b_2 < +\infty, 1 < p < +\infty$ and the function$u(x_1, x_2)$ be defined on$\Delta = (a_1, a_2; b_1, b_2)$, moreover let this function be measurable.

We make the following notations

$L_p^{loc}(\Delta) = \{u: \forall \xi_i, \eta_i > 0, i = 1,2, \xi_1 + \eta_1 \leq a_2 - a_1 = l_1, \xi_2 + \eta_2 \leq b_2 - b_1 = l_2, u \in L_p[a_1 + \xi_1 a_2 - \eta_1; b_1 + \xi_2, b_2 - \eta_2]\}$,

$L_p^{loc}(a_1, b_1) = \{u: \forall \xi_i \in (0, l_i], i = 1,2, u \in L_p[a_1 + \xi_1, a_2; b_1 + \xi_2, b_2]\}$,

$L_p^{loc}(a_2, b_1) = \{u: \forall \xi_i \in (0, l_i], i = 1,2, u \in L_p[a_1, a_2 - \xi_1; b_1 + \xi_2, b_2]\}$,

$L_p^{loc}(a_1, b_2) = \{u: \forall \xi_i \in (0, l_i], i = 1,2, u \in L_p[a_1 + \xi_1, a_2; b_1, b_2 - \xi_2]\}$,

$L_p^{loc}(a_2, b_2) = \{u: \forall \xi_i \in (0, l_i], i = 1,2, u \in L_p[a_1, a_2 - \xi_1; b_1, b_2 - \xi_2]\}$.

For the function$u_{ij} \in L_p^{loc}(a_i, b_j)$ $(i, j = 1,2)$we introduce the characteristic

$$\Omega_p^{11}(u_{11}, \xi_1, \xi_2,) = \left(\int_{a_1+\xi_1}^{a_2} \int_{b_2+\xi_2}^{b_2} |u_{11}(x_1, x_2)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}},$$

$$\varpi_p^{11}(u_{11}, \delta_1, \xi_1, \xi_2) = \sup_{\square_1 \in E_1} \left(\int_{a_1+\xi_1}^{a_2-\xi_1-\square_1} \int_{b_1+\xi_2}^{b_2} |u_{11}(x_1 + \square_1, x_2) - u_{11}(x_1, x_2)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}},$$

$$\overline{\varpi}_p^{11}(u_{11}, \xi_1, \delta_2, \xi_2) = \sup_{\square_2 \in E_2} \left(\int_{a_1+\xi_1}^{a_2} \int_{b_1+\xi_2}^{b_2-\xi_2-\square_2} |u_{11}(x_1, x_2 + \square_2) - u_{11}(x_1, x_2)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}},$$

$$\omega_p^{11}(u_{11}, \delta_1, \xi_1, \delta_2, \xi_2) = \sup_{\square_1 \in E_1 \square_2 \in E_2} \left(\int_{a_1+\xi_1}^{a_2-\xi_1-\square_1} \int_{b_1+\xi_2}^{b_2-\xi_2-\square_2} |\Delta u_{11}(x_1 + \square_1, x_1, x_2 + \square_2, x_2)|^p \, dx_1 dx_2 \right)^{\frac{1}{p}},$$

where $\delta_i > 0, \mathrm{E}_i = \{h_i: 0 < h_i \leq \min\{\delta_i, l_i - \xi_i\}\}. i = 1,2, \Delta u_{11}(x_1 + h_1, x_1, x_2 + h_2, x_2) = u_{11}(x_1 + h_1, x_2 + h_2) - u_{11}(x_1 + h_1, x_2) - u_{11}(x_1, x_2 + h_2) + u_{11}(x_1, x_2)$.

Let$u \in L_p^{loc}(\Delta)$and $u(x_1, x_2) = \sum_{i,j=1,2}^{2} u_{ij}(x_1, x_2)$, where$u_{ij} \in L_p^{loc}(a_i, b_j)$

$(i, j = 1,2)$.

Then it is obvious that the function

$$\tilde{u}(x_1, x_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \frac{u(s_1, s_2)}{(s_1 - x_1)(s_2 - x_2)} ds_1 ds_2$$

can be represented in the form

$$\tilde{u}(x_1, x_2) = \sum_{i,j=1,2}^{2} \tilde{u}_{ij}(x_1, x_2)$$

Using[8], [15], [17], [18], it is proved the following theorem.

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Theorem 1. Let $u_{ij} \in L_p^{loc}(a_i, b_j), \xi_{ij}^k \in (0, l_k), (i, j, k = 1,2)$.Thenfromconvergenceofcorrespondenceintegralsit holds the following inequality

$$\Omega_p^{ij}(\tilde{u}_{ij}, \xi_{ij}^1, \xi_{ij}^2) \leq\leq C_p[\int_0^{\frac{l_1}{2}}\int_0^{\frac{l_2}{2}} \frac{\Omega_p^{ij}(u_{ij}, t_1, t_2)}{(t_1t_2)^{\frac{1}{p}}(t_1+\xi_{ij}^1)^{\frac{1}{q}}(t_2+\xi_{ij}^2)^{\frac{1}{q}}} dt_1dt_2 +$$

$$\int_0^{\frac{l_1}{2}}\int_0^{\frac{\xi_{ij}^2}{2}} \frac{\bar{\omega}_p^{ij}\left(u_{ij}, t_1, t_2, \frac{\xi_{ij}^2}{2}\right)}{t_1^{\frac{1}{p}}(t_1+\xi_{ij}^1)^{\frac{1}{q}}t_2} dt_1dt_2 + \int_0^{\frac{\xi_{ij}^1}{2}}\int_0^{\frac{l_2}{2}} \frac{\bar{\bar{\omega}}_p^{ij}\left(u_{ij}, t_1, \frac{\xi_{ij}^1}{2}, t_2\right)}{t_2^{\frac{1}{p}}(t_2+\xi_{ij}^2)^{\frac{1}{q}}t_1} dt_1dt_2 + \int_0^{\frac{\xi_{ij}^1}{2}}\int_0^{\frac{\xi_{ij}^2}{2}} \frac{\omega_p^{ij}\left(u_{ij}, t_1, \frac{\xi_{ij}^1}{2}, t_2, \frac{\xi_{ij}^2}{2}\right)}{t_1t_2} dt_1dt_2$$

$$+\beta_1(\xi_{ij}^1)\beta_2(\xi_{ij}^2)\Omega_p^{k_1k_2}(u_{ij}, \frac{l_1}{3}, \frac{l_2}{3}) ln\frac{l_1}{\xi_{ij}^1} ln\frac{l_2}{\xi_{ij}^2}],$$

where

$$.\beta_1(x) = \begin{cases} 1, & if x \in \left(0, \frac{l_k}{3}\right], \\ 0, & if x \bar{\in} \left(0, \frac{l_k}{3}\right], \end{cases}$$

(i,j,$k_1, k_2 = 1,2, i \neq k_1, j \neq k_2$)

Moreover, itisobtainedestimates$\bar{\omega}_p^{ij}(\tilde{u}_{ij}, \delta_1, \xi_{ij}^1, \xi_{ij}^2)$, $\bar{\bar{\omega}}_p^{ij}(\tilde{u}_{ij}, \xi_{ij}^1, \delta_2, \xi_{ij}^2)$,

$\omega_p^{ij}(\tilde{u}_{ij}, \delta_1, \xi_{ij}^1, \delta_2, \xi_{ij}^2)$. We denote by G the set of positive functions

$(\varphi(\xi_1, \xi_2,), \bar{\psi}(\delta_1, \xi_1, \xi_2,), \bar{\bar{\psi}}(\xi_1, \delta_2, \xi_2), \psi(\delta_1, \xi_1, \delta_2, \xi_2))$,

definedfor$0 < \xi_i < l_i, \delta_{i,} > 0, i = 1,2$,andsuchthatthe functions $\varphi, \bar{\psi}, \bar{\bar{\psi}}, \psi$ almostdecreasinginξ_1, ξ_2 (uniformly by other variables) , $\bar{\psi}, \bar{\bar{\psi}}, \psi$almost increasing in δ_1, δ_2 (uniformly by other variables)

$$\frac{\bar{\psi}(\delta_1, \xi_1, \xi_2)}{\delta_1}, \frac{\bar{\bar{\psi}}(\xi_1, \delta_2, \xi_2)}{\delta_2}, \frac{\psi(\delta_1, \xi_1, \delta_2, \xi_2)}{\delta_1}, \frac{\psi(\delta_1, \xi_1, \delta_2, \xi_2)}{\delta_2},$$

almostdecreasinginδ_1, δ_2 (uniformly by other variables)

$$\bar{\psi}(\delta_1, \xi_1, \xi_2), \bar{\bar{\psi}}(\xi_1, \delta_2, \xi_2), \psi(\delta_1, \xi_1, \delta_2, \xi_2) \to 0$$

for$\delta_1, \delta_2 \to 0$.

Let$\varphi, \bar{\psi}, \bar{\bar{\psi}}, \psi \in G$.Denoteby$H_{\varphi,\bar{\psi},\bar{\bar{\psi}},\psi}^{p,a_1,b_1}$the set of functions from$L_p^{loc}(a_1, b_1)$ such that there exists constant$c_i > 0$ $(i = 1,4)$ and

$$\Omega_p^{11}(u_{ij}, \xi_1, \xi_2) \leq c_1\varphi(\xi_1, \xi_2),$$

$$\bar{\omega}_p^{11}(u_{ij}, \delta_1, \xi_1, \xi_2) \leq c_2\bar{\psi}(\delta_1, \xi_1, \xi_2),$$

$$\bar{\bar{\omega}}_p^{11}(u_{ij}, \xi_1, \delta_2, \xi_2) \leq c_3\bar{\bar{\psi}}(\xi_1, \delta_2, \xi_2),$$

$\omega_p^{11}(u_{ij}, \delta_1, \xi_1, \delta_2, \xi_2) \leq c_4\psi(\delta_1, \xi_1, \delta_2, \xi_2)$.

Theset$H_{\varphi,\bar{\psi},\bar{\bar{\psi}},\psi}^{p,a_1,b_1}$bynorm$\|u\|_{H_{\varphi,\bar{\psi},\bar{\bar{\psi}},\psi}^{p,a_1,b_1}} = \max\{c_1, c_2, c_3, c_4\}$is a Banach space.

ByG_0 we denote the set of function $\varphi, \bar{\psi}, \bar{\bar{\psi}}, \psi \in G$, such that for $\forall\xi_i \in (0, l_i]$ the following integrals are convergent

$$\int_0^{l_1}\int_0^{l_2} \frac{\varphi(t_1, t_2)}{(t_1t_2)^{\frac{1}{p}}(t_1+\xi_1)^{\frac{1}{q}}(t_2+\xi_2)^{\frac{1}{q}}} dt_1dt_2, \int_0^{l_1}\int_0^{l_2} \frac{\bar{\psi}\left(t_1\frac{\xi_1}{2}, t_2\right)}{(t_1t_2)^{\frac{1}{p}}(t_2+\xi_2)^{\frac{1}{q}}} dt_1dt_2,$$

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$$\int_0^{l_1}\int_0^{l_2}\frac{\bar{\bar{\psi}}\left(t_1,t_2,\frac{\xi_2}{2}\right)}{(t_1t_2)^{\frac{1}{p}}(t_1+\xi_1)^{\frac{1}{q}}}dt_1dt_2,\int_0^{l_1}\int_0^{l_2}\frac{\varphi\left(t_1,\frac{\xi_2}{2},t_2,\frac{\xi_2}{2}\right)}{(t_1t_2)^{\frac{1}{p}}}dt_1dt_2,$$

Now, wedeterminedby$G_0H_P$the set of positive functions $\varphi(\xi_1,\xi_2),\bar{\psi}(\xi_1,\delta_2,\xi_2)$, $\bar{\bar{\psi}}(\delta_1,\xi_1,\xi_2)$, $\psi(\delta_1,\xi_1,\delta_2,\xi_2,)$satisfying the following conditions:

$$\int_0^{l_1}\int_0^{l_2}\frac{\varphi(t_1,t_2)}{(t_1t_2)^{\frac{1}{p}}(t_1+\xi_1)^{\frac{1}{q}}(t_2+\xi_2)^{\frac{1}{q}}}dt_1dt_2=0(\varphi(\xi_1,\xi_2)),$$

$$\delta_1\int_0^{l_1}\int_0^{l_2}\frac{\bar{\psi}\left(t_1,\frac{\xi_1}{2},t_2\right)}{(t_1t_2)^{\frac{1}{p}}(t_2+\xi_2)^{\frac{1}{q}}}dt_1dt_2=0(\bar{\psi}(\delta_1,\xi_1,\xi_2)),$$

$$\delta_2\int_0^{l_1}\int_0^{l_2}\frac{\bar{\bar{\psi}}\left(t_1,t_2\frac{\xi_2}{2}\right)}{(t_1t_2)^{\frac{1}{p}}(t_1+\xi_1)^{\frac{1}{q}}}dt_1dt_2=0(\bar{\bar{\psi}}(\xi_1,\delta_2,\xi_2)),$$

$$\delta_1\delta_2\int_0^{l_1}\int_0^{l_2}\frac{\psi\left(t_1,\frac{\xi_1}{2},t_2,\frac{\xi_2}{2}\right)}{(t_1t_2)^{\frac{1}{p}}}dt_1dt_2=0(\psi(\delta_1,\xi_1,\delta_2,\xi_2)),$$

$$\frac{\delta_1}{\delta_1+\xi_1}\varphi(\xi_1,\xi_2)=0\left(\bar{\psi}(\delta_1,\xi_1,\xi_2)\right),\frac{\delta_1}{\delta_2+\xi_2}\varphi(\xi_1,\xi_2)=0(\bar{\bar{\psi}}(\xi_1,\delta_2,\xi_2)),$$

$$\frac{\delta_1}{\delta_1+\xi_1}\frac{\delta_1}{\delta_2+\xi_2}\varphi(\xi_1,\xi_2)=0(\psi(\delta_1,\xi_1,\delta_2,\xi_2)),$$

where the constants in expression"0"do not depend on $\delta_i,\xi_i(i=1,2)$.

We also denote

$$H^{p,a_i,b_j}_{ln\frac{2l_1}{\xi_{ij}^1}ln\frac{2l_2}{\xi_{ij}^2},\frac{\delta_1}{\delta_1+\xi_{ij}^1}ln\frac{2l_2}{\xi_{ij}^2},\frac{\delta_2}{\delta_2+\xi_{ij}^2}ln\frac{2l_1}{\xi_{ij}^1},\frac{\delta_1}{\delta_1+\xi_{ij}^1}\frac{\delta_2}{\delta_2+\xi_{ij}^2}}=M_{IJ}^P$$

Theorem 2. If$\left(\varphi_{IJ},\bar{\psi}_{IJ},\bar{\bar{\psi}}_{IJ},\psi_{IJ}\right)\in G_0H_P$.Then operator $\tilde{u}$maps the space $\sum_{I,J=1}^2 H^{p,a_I,b_J}_{\varphi_{IJ},\bar{\psi}_{IJ},\bar{\bar{\psi}}_{IJ},\psi_{IJ}}$itself and is boundedby M_{IJ}^P (i,j=1,2).

We note that the proof of this last Theorem 2 comes from the proof of Theorem 1 and by definition of the sets of G_0H_P.

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