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Some Finite Summation Formulas Involving Multivariable Hypergeometric Polynomials

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Abstract:

The main purpose of this paper is to present a family of finite summation formulas and to apply it in order to derive several functional relationships involving various multivariable hypergeometric polynomials and the Gauss hypergeometric function. A number of special and limit cases of these functional relationships are also considered.

Keywords: Finite summation formulas, multivariable hypergeometric polynomials, functional relationships, Gauss hypergeometric function, orthogonal and biorthogonal polynomials, multinomial theorem.

INTRODUCTION

In terms of a bounded multiple sequence $\Omega(k_1,\ldots,k_r)$ of essentially arbitrary (real or complex) parameters, let

$$\Theta^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) := \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1k_1} \cdots (-n_r)_{m_rk_r} \cdot \Omega(k_1,\ldots,k_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!} \qquad (1.1)$$

$$\left(n_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ m_j \in \mathbb{N};\ j = 1,\ldots,r\right),$$

where $[\kappa]$ denotes the greatest integer in $\kappa \in$ R and $(\lambda)k$ is the Pochhammer symbol (or, more precisely, the *shifted* factorial, since $(1)k = k!$ $(k \in$ N0)) defined, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0; \lambda \neq 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}), \end{cases} \qquad (1.2)$$

N being the set of *positive* integers. For different choices of the multiple sequence $\{\Omega(k1,\ldots,kr)\}$ *and* with

$mj = 1$ $(j = 1,\ldots,r)$,

the multivariable polynomials [*cf*. Equation (1.1)]

$$\Theta^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$$

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would readily yield, as special cases, various classes of orthogonal and biorthogonal polynomials associated with hypergeometric functions of two and more variables (see, for details, [1], [6], [7], and [12] to [18]).

MAIN PART

Motivated essentially by these and sundry other occurrences of special multivariable hypergeometric polynomials in the mathematical and physical sciences literature, we first propose to derive here a family of finite summation formulas involving the polynomials defined by (1.1) and then show how this general result can be applied in order to deduce several functional relationships between various multivariable hypergeometric polynomials and the Gauss hypergeometric function which corresponds to the familiar special case

$p - 1 = q = 1$

of the generalized hypergeometric pFq function with p numerator and q denominator parameters, defined by

$$ {}_pF_q\left[(\alpha_p);(\beta_q);z\right] = {}_pF_q\left[\begin{matrix}(\alpha_p); \\ (\beta_q);\end{matrix}\; z\right] $$

$$:= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \tag{1.3} $$

$$ \left(p, q \in \mathbb{N}_0;\; p \leqq q+1;\; p \leqq q \;\text{ and }\; |z| < \infty;\right. $$

$$ \left. p = q+1 \;\text{ and }\; |z| < 1;\; p = q+1,\; |z| = 1, \;\text{ and }\; \Re(\Xi) > 0\right), $$

where (*and throughout this paper*) we find it to be convenient to abbreviate the p-parameter array:

$\alpha 1, \ldots, \alpha p$ ($p \in \mathbb{N}$)

by (αp), the array being empty when $p = 0$, with similar interpretations for (βq), *et cetera*, and

$$ \Xi := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{q} \alpha_j \qquad \left(\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right). $$

We begin by recalling the multinomial theorem in the form (*cf*., *e.g*., [3, p. 13, Eq. 2.3 (9)]):

$$ \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r} = (x_1 + \cdots + x_r)^n \tag{2.1} $$

$$ \left(n, n_j \in \mathbb{N}_0;\; j = 1, \ldots, r;\; r \in \mathbb{N} \setminus \{1\}\right), $$

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where, *and in what follows*, since

$$\binom{n}{n_1,\ldots,n_r} := \frac{n!}{n_1!\cdots n_r!}. \tag{2.2}$$

$$(-n_j)_{m_j k_j} = (-1)^{m_j k_j} \frac{n_j!}{(n_j - m_j k_j)!} \tag{2.3}$$

$$\left(0 \leqq k_j \leqq [n_j/m_j]\,;\; j = 1,\ldots,r\right),$$

by virtue of the definition (1.2), we can make use of the multinomial theorem (2.1) in conjunction with the definition (1.1) to show that

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} \Theta^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\, t_1^{n_1}\cdots t_r^{n_r}$$

$$= T^n \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r \leqq n} (-n)_{m_1k_1+\cdots+m_rk_r}\, \Omega(k_1,\ldots,k_r)$$

$$\cdot \frac{\{x_1 (t_1/T)^{m_1}\}^{k_1}}{k_1!} \cdots \frac{\{x_r (t_r/T)^{m_r}\}^{k_r}}{k_r!} \tag{2.4}$$

$$\left(T := t_1 + \cdots + t_r;\; n, n_j \in \mathbb{N}_0;\; m_j \in \mathbb{N};\; j = 1,\ldots,r\right).$$

With a view to applying the general finite summation formula (2.4) to the following familiar *special* case of the (Srivastava-Daoust) generalized Lauricella functions, defined by (*cf*., *e.g*., [9, p. 38, Eq. 1.4 (24)])

$$F^{p:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r}\left[\begin{matrix}(\alpha_p):\left(\gamma'_{p_1}\right);\cdots;\left(\gamma^{(r)}_{p_r}\right); \\ (\beta_q):\left(\delta'_{q_1}\right);\cdots;\left(\delta^{(r)}_{q_r}\right);\end{matrix}\; z_1,\ldots,z_r\right]$$

$$:= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{k_1+\cdots+k_r} \prod_{j=1}^{p_1}\left(\gamma'_j\right)_{k_1}\cdots\prod_{j=1}^{p_r}\left(\gamma^{(r)}_j\right)_{k_r}}{\prod_{j=1}^{q}(\beta_j)_{k_1+\cdots+k_r} \prod_{j=1}^{q_1}\left(\delta'_j\right)_{k_1}\cdots\prod_{j=1}^{q_r}\left(\delta^{(r)}_j\right)_{k_r}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}, \tag{2.5}$$

whenever the multiple hypergeometric series in (2.5) converges or terminates, we conveniently set

$m1 = \cdots = mr = 1$,

and we find from (2.4) that

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} F^{p:p_1+1;\cdots;p_r+1}_{q:q_1+1;\cdots;q_r+1} \left[\begin{array}{l} (\alpha_p) : -n_1, \left(\gamma'_{p_1}\right); \cdots ; -n_r, \left(\gamma^{(r)}_{p_r}\right) \\ (\beta_q) : \quad \left(\delta'_{q_1}\right); \cdots ; \left(\delta^{(r)}_{q_r}\right); \end{array} x_1, \ldots, x_r \right] t_1^{n_1} \cdots t_r^{n_r}$$

$$= T^n \; F^{p+1:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r} \left[\begin{array}{r} -n, (\alpha_p) : \left(\gamma'_{p_1}\right); \cdots ; \left(\gamma^{(r)}_{p_r}\right); \\ (\beta_q) : \left(\delta'_{q_1}\right); \cdots ; \left(\delta^{(r)}_{q_r}\right); \end{array} \frac{x_1 t_1}{T}, \ldots, \frac{x_r t_r}{T} \right] \tag{2.6}$$

$$\left(T := t_1 + \cdots + t_r;\ n, n_j \in \mathbb{N};\ j = 1, \ldots, r\right).$$

Next, by appealing appropriately to the multiple series identity (*cf.* [8]; see also [9, p. 39]):

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \omega\left(n_1 + \cdots + n_r\right) \left(\lambda_1\right)_{n_1} \cdots \left(\lambda_r\right)_{n_r} \frac{z^{n_1}}{n_1!} \cdots \frac{z^{n_r}}{n_r!}$$

$$= \sum_{n=0}^{\infty} \omega(n) \left(\lambda_1 + \cdots + \lambda_r\right)_n \frac{z^n}{n!} \tag{2.7}$$

and its multivariable hypergeometric form:

$$F^{p:1;\cdots;1}_{q:0;\cdots;0} \left[\begin{array}{l} (\alpha_p) : \lambda_1; \cdots ; \lambda_r; \\ (\beta_q) : —; \cdots ; —; \end{array} z, \ldots, z \right]$$

$$= {}_{p+1}F_q \left[\begin{array}{r} (\alpha_p), \lambda_1 + \cdots + \lambda_r; \\ (\beta_q); \end{array} z \right], \tag{2.8}$$

the second members of (2.4) and (2.6) can be simplified considerably in the special cases:

$$\Omega(k_1,\ldots,k_r)=(\lambda_1)_{k_1}\cdots(\lambda_r)_{k_r}\,\omega(k_1+\cdots+k_r),\quad m_j=m,\quad \text{and}\quad x_j=\left(\frac{\tau}{t_j}\right)^m \tag{2.9}$$

$$(m\in\mathbb{N};\ k_j\in\mathbb{N}_0;\ \lambda_j\in\mathbb{C};\ j=1,\ldots,r)$$

And

$$p_j-1=q_j=0\quad\left(\gamma_{p_j}^{(j)}=\lambda_j\right)\quad\text{and}\quad x_jt_j=\tau\quad(j=1,\ldots,r), \tag{2.10}$$

respectively. We thus find from (2.4) that

$$\sum_{n_1+\cdots+n_r=n}\binom{n}{n_1,\ldots,n_r}\Phi_{n_1,\ldots,n_r}^{\lambda_1,\ldots,\lambda_r}\left(m;\frac{\tau}{t_1},\ldots,\frac{\tau}{t_r}\right)t_1^{n_1}\cdots t_r^{n_r}$$

$$=T^n\sum_{k=0}^{[n/m]}(-n)_{mk}(\lambda_1+\cdots+\lambda_r)_k\,\omega(k)\frac{(\tau/T)^{mk}}{k!} \tag{2.11}$$

$$(T:=t_1+\cdots+t_r;\ m\in\mathbb{N};\ n,n_j\in\mathbb{N}_0;\ \lambda_j\in\mathbb{C};\ j=1,\ldots,r),$$

where [*cf*. Equation (1.1)]

$$\Phi_{n_1,\ldots,n_r}^{\lambda_1,\ldots,\lambda_r}(m;x_1,\ldots,x_r):=\sum_{k_1=0}^{[n_1/m]}\cdots\sum_{k_r=0}^{[n_r/m]}(-n_1)_{mk_1}(\lambda_1)_{k_1}\cdots(-n_r)_{mk_r}(\lambda_r)_{k_r}$$

$$\cdot\,\omega(k_1+\cdots+k_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!} \tag{2.12}$$

$$(n_j\in\mathbb{N}_0;\ \lambda_j\in\mathbb{C};\ j=1,\ldots,r;\ m\in\mathbb{N})$$

in terms of a bounded sequence $\{\omega(n)\}_{n=0}^{\infty}$ of essentially arbitrary (real or complex) parameters. Furthermore, under the constraints given by (2.10), the finite summation formula (2.6) similarly yields

$$\sum_{n_1+\cdots+n_r=n}\binom{n}{n_1,\ldots,n_r}F_{q:0;\cdots;0}^{p:2;\cdots;2}\left[\begin{array}{l}(\alpha_p):-n_1,\lambda_1;\cdots;-n_r,\lambda_r;\\(\beta_q):\underline{\quad\quad};\cdots;\underline{\quad\quad};\end{array}\frac{\tau}{t_1},\ldots,\frac{\tau}{t_r}\right]t_1^{n_1}\cdots t_r^{n_r}$$

$$=T^n\ {}_{p+2}F_q\left[\begin{array}{r}-n,\lambda_1+\cdots+\lambda_r,(\alpha_p);\\(\beta_q);\end{array}\frac{\tau}{T}\right] \tag{2.13}$$

$$(T:=t_1+\cdots+t_r;\ n,n_j\in\mathbb{N}_0;\ \lambda_j\in\mathbb{C};\ j=1,\ldots,r).$$

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